STABILITY OF ROTATIONAL MOTION OF A GEOMETRICALLY SYMMETRICAL ARTIFICIAL SATELLITE OF THE SUN IN THE FIELD OF LIGHT PRESSURE FORCES

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Considered is the librational motion of an artificial satellite of the sun relative to its center of inertia in a parallel light flow. Derived are the formulas for the moment of light pressure forces acting upon the body bounded by a surface of revolution. The obtained expressions permit the investigation of rotational motion stability of a geometrically symmetrical satellite of the sun whose "tracking" axis of symmetry is directed to the sun.

It was shown in [1] that the principal force **F** and the principal moment **M** of light pressure forces acting on a body of arbitrary form with a uniform surface reflection coefficient in the parallel light flow field are

$$\mathbf{F} = (1-\varepsilon)\,\mathbf{F}^+ + \varepsilon\mathbf{F}^- = \mathbf{F}^+ + \varepsilon\,(\mathbf{F}^- - \mathbf{F}^+) \qquad (1)$$

$$\mathbf{M} = (1-\varepsilon)\,\mathbf{M}^+ + \varepsilon\mathbf{M}^- = \mathbf{M}^+ + \varepsilon\,(\mathbf{M}^- - \mathbf{M}^+)$$

Here $\mathbf{F}^+$, $\mathbf{F}^-$, $\mathbf{M}^+$, $\mathbf{M}^-$ are the vectors of forces and torques acting on the body if its surface were completely absorbing $(\mathbf{F}^+, \mathbf{M}^+)$ or completely reflecting $(\mathbf{F}^-, \mathbf{M}^-)$, and ε is the reflection coefficient.

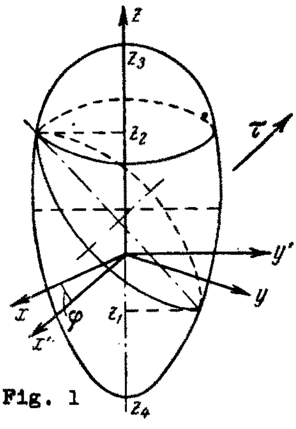

Fig. 1

Similarly,

$$\mathbf{F}^+ = -h_0 \int\limits_{(s_1)} \boldsymbol{\tau}\,(\boldsymbol{\tau}\cdot\mathbf{n})\,dS = -h_0\boldsymbol{\tau} \int\limits_{(s_1)} \boldsymbol{\tau}\cdot\mathbf{n}\,dS \qquad (2)$$

$$\mathbf{M}^+ = -h_0 \int\limits_{(s_1)} [\mathbf{R}\times\boldsymbol{\tau}]\,(\boldsymbol{\tau}\cdot\mathbf{n})\,dS = h_0\boldsymbol{\tau}\times \int\limits_{(s_1)} \mathbf{R}\,(\boldsymbol{\tau}\cdot\mathbf{n})\,dS$$

$$\mathbf{F}^- = -2h_0 \int\limits_{(s_1)} \mathbf{n}\,(\boldsymbol{\tau}\cdot\mathbf{n})^2\,dS$$

$$\mathbf{M}^- = -2h_0 \int\limits_{(s_1)} \mathbf{R}\times\mathbf{n}\,(\boldsymbol{\tau}\cdot\mathbf{n})^2\,dS \qquad (3)$$

Here h_0 is a quantity inversely proportional to the square of the distance from the light source which has the units of pressure, $\mathbf{n}$ is a unit vector of the external normal to the surface, $\mathbf{R}$ is the radius vector of an arbitrary point on the surface of the body, $\boldsymbol{\tau}$ is a unit vector directed opposite to the light flow, (s_1) is the illuminated part of the surface which is defined by Equation

$$\boldsymbol{\tau}\cdot\mathbf{n} \geqslant 0 \tag{4}$$

Let us apply Formulas (2 and 3) for computation of the moment acting on a body bounded by a surface of revolution, equation of which in Cartesian coordinates has the form (Fig.1)

$$x^2 + y^2 = R^2(z) = f(z) \tag{5}$$

where $R(z)$ is the cross-sectional radius and z is the axis of symmetry. A body bounded by such a surface will be termed the geometrically symmetric body. For simplicity of exposition, we will assume that the surface contains no planes or cylindrical and conical sections although, as will become apparent subsequently, the integral computation method will also be valid for these cases as well. The essential assumption is only regarding the convexity of the surface which means that the curve $f(z)$ has a single maximum. The coordinate for the maximum of the function $f(z)$ will be denoted by z^*, and the cross-section corresponding to the coordinate $z = z^*$ will be termed the "midship" cross-section (maximum cross-section).

The projections of the unit vector $\mathbf{n}$ along the external normal on the x, y and z axes are

$$n_x = \frac{x}{\sqrt{f(z) + {}^1/_4\,[f'(z)]^2}} = \frac{x}{R(z)\sqrt{1 + [R'(z)]^2}}$$

$$n_y = \frac{y}{\sqrt{f(z) + {}^1/_4\,[f'(z)]^2}} = \frac{y}{R(z)\sqrt{1 + [R'(z)]^2}} \qquad \left(f'(z) = \frac{df}{dz}\right) \tag{6}$$

$$n_z = -\frac{{}^1/_2\, f'(z)}{\sqrt{f(z) + {}^1/_4\,[f'(z)]^2}} = -\frac{R'(z)}{\sqrt{1 + [R'(z)]^2}} \qquad \left(R'(z) = \frac{dR}{dz}\right)$$

Denote, further, by a_0, b_0, c_0, the projections of the unit vector $\boldsymbol{\tau}$ on the x, y, z axes respectively. A transfer from the x, y, z axes to the x', y', z axes is accomplished by means of a rotation about the z-axis such that the y'-axis would coincide with the projection of the unit vector $\boldsymbol{\tau}$ on the surface $x'y'$; we pass from the coordinates x', y', z to the surface coordinates z, φ; the angle φ is measured from the x'-axis. Then

$$\begin{aligned} x &= x'\frac{b_0}{\sqrt{1 - c_0^2}} + y'\frac{a_0}{\sqrt{1 - c_0^2}}, \qquad & x' &= R(z)\cos\varphi \\ y &= -x'\frac{a_0}{\sqrt{1 - c_0^2}} + y'\frac{b_0}{\sqrt{1 - c_0^2}}, \qquad & y' &= R(z)\sin\varphi \end{aligned} \tag{7}$$

$$\boldsymbol{\tau}\cdot\mathbf{n} = \frac{\sqrt{1 - c_0^2}\sin\varphi - c_0 R'(z)}{\sqrt{1 + [R'(z)]^2}}, \qquad dS = R(z)\sqrt{1 + [R'(z)]^2}\, d\varphi\, dz \tag{8}$$

The equation for the terminator $\boldsymbol{\tau}\cdot\mathbf{n} = 0$ is of the form (*)

$$\sqrt{1 - c_0^2}\sin\varphi - c_0 R'(z) = 0 \tag{9}$$

which determines the integration regions in (2) and (3). Noting that the representation of the unit vector $\boldsymbol{\tau}$ in Fig.1 corresponds to the case of $c_0 > 0$ and assuming inner integration along φ, we find that the integration region consists of two sub-regions. In the first sub-region the limits of integration along z are from $z_1(c_0)$ to $z_2(c_0)$, while along φ they are from $\varphi^*(z, c_0)$ to $\pi - \varphi^*(z, c_0)$, where

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*) The terminator is the line dividing the illuminated and dark side of the satellite.

$$\sin\varphi^* = \frac{c_0 R'(z)}{\sqrt{1-c_0^2}} \tag{10}$$

and where the determination of z_1 and z_2 is obtained with the aid of Equation

$$[R'(z)]^2 = \frac{1-c_0^2}{c_0^2} \qquad (z_2 > z_1) \tag{11}$$

The limits of integration in the second sub-region are $z_2(c_0)$ to z_3 along z and $-\frac{1}{2}\pi$ to $\frac{3}{2}\pi$ along φ.

In the case $c_0 < 0$, i.e. when the body is illuminated from "below" (from the negative direction of the z-axis), the limits of outer integration in the second sub-region are, naturally, from z_4 to z_1. Note that in the general case of an arbitrary convex surface of revolution the simultaneous existence of both sub-regions is not necessary (the cone, side surface of a cylinder). Let us denote

$$\mathbf{M}^+ = h_0 \boldsymbol{\tau} \times \mathbf{I} \tag{12}$$

where $\mathbf{I}$ is a vector quantity the projections of which on the x, y, z axes are

$$I_x = \int_{(s_1)} x(\boldsymbol{\tau}\cdot\mathbf{n})\,ds = \int_{z_1}^{z_2}\int_{\varphi^*}^{\pi-\varphi^*} G(z,\varphi)\,d\varphi\,dz + \int_{z_2}^{z_3}\int_{-1/2\pi}^{3/2\pi} G(z,\varphi)\,d\varphi\,dz \tag{13}$$

$$I_y = \int_{(s_1)} y(\boldsymbol{\tau}\cdot\mathbf{n})\,ds = \int_{z_1}^{z_2}\int_{\varphi^*}^{\pi-\varphi^*} H(z,\varphi)\,d\varphi\,dz + \int_{z_2}^{z_3}\int_{-1/2\pi}^{3/2\pi} H(z,\varphi)\,d\varphi dz \tag{14}$$

$$I_z = \int_{(s_1)} z(\boldsymbol{\tau}\cdot\mathbf{n})\,ds = \int_{z_1}^{z_2}\int_{\varphi^*}^{\pi-\varphi^*} zF(z,\varphi)\,d\varphi\,dz + \int_{z_2}^{z_3}\int_{-1/2\pi}^{3/2\pi} zF(z,\varphi)\,d\varphi\,dz \tag{15}$$

In Formulas (13) to (15) the following is used

$$\begin{aligned} G(z,\varphi) &= (1-c_0^2)^{-1/2} R(z)\,F(z)\,(b_0\cos\varphi + a_0\sin\varphi) \\ H(z,\varphi) &= (1-c_0^2)^{-1/2} R(z)\,F(z)\,(b_0\sin\varphi - a_0\cos\varphi) \\ F(z,\varphi) &= R(z)\,[\sqrt{1-c_0^2}\sin\varphi - c_0 R'(z)] \end{aligned} \tag{16}$$

The integral

$$\int_{\varphi^*}^{\pi-\varphi^*} \sin^n\varphi\cos\varphi\,d\varphi = \frac{1}{n+1}\sin^{n+1}\varphi\Big|_{\varphi^*}^{\pi-\varphi^*} \equiv 0 \qquad (n=0,1,2,\ldots) \tag{17}$$

Therefore, in the expression for I_x all terms with the b_0 coefficient are identically zero, while in the expression for I_y, the a_0 terms are identically zero. Consequently,

$$I_x = \frac{a_0}{\sqrt{1-c_0^2}} J, \qquad I_y = \frac{b_0}{\sqrt{1-c_0^2}} J \tag{18}$$

$$I_z = 2\sqrt{1-c_0^2}\int_{z_1}^{z_2} zR(z)(\cos\varphi^* + \varphi^*\sin\varphi^*)\,dz + \frac{\pi c_0}{2}[z_2 f(z_2) + z_1 f(z_1)] + \\ + \pi c_0\left[\frac{1}{2}\int_{z_1}^{z_2} f(z)\,dz + \int_{z_2}^{z_3} f(z)\,dz\right] \tag{19}$$

where

$$J = \sqrt{1-c_0^2}\left\{\pi\left[\frac{1}{2}\int_{z_1}^{z_2} f(z)\,dz + \int_{z_2}^{z_3} f(z)\,dz\right] - \int_{z}^{z_2} f(z)(\varphi^* + \sin\varphi^*\cos\varphi^*)\,dz\right\}$$

Having computed the product $\tau \times \mathbf{I}$, taking into account (18) and (19), and carrying out analogous calculations for $c_0 < 0$ we get

$$M_x^+ = h_0 b_0 \Phi^+(c_0), \qquad M_y^+ = -h_0 a_0 \Phi^+(c_0), \qquad M_z^+ = 0 \tag{20}$$

Here

$$\Phi^+(c_0) = 2\sqrt{1-c_0^2}\int_{z_1}^{z_2} zR(z)(\cos\varphi^* + \varphi^*\sin\varphi^*)\,dz +$$

$$+ c_0 \int_{z_1}^{z_2} f(z)(\varphi^* + \sin\varphi^*\cos\varphi^*)dz + \frac{\pi|c_0|}{2}[z_2 f(z_2) + z_1 f(z_1)]$$

will be an even function of the quantity c_0.

It can be easily noted, also, that in the case when the surface of the body has double symmetry, i.e. $R(z) = R(-z)$, $z_2 = -z_1$ (for example, ellipsoid of revolution), the function $\Phi^+(c_0)$ is identically zero.

Let us pass to the calculation of the quantity $\mathbf{M}^-$. In the projections on the axes x, y, z, this quantity becomes after substitution of Expression (6)

$$M_x^- = 2h_0 \int_{(s_1)} y(\tau\cdot\mathbf{n})^2 \frac{z + R'(z)R(z)}{\sqrt{1+[R'(z)]^2}}\,dz,$$

$$M_y^- = -2h_0 \int_{(s_1)} x(\tau\cdot\mathbf{n})^2 \frac{z + R'(z)R(z)}{\sqrt{1+[R'(z)]^2}}\,dz, \qquad M_z^- = 0 \tag{21}$$

Substituting (7) to (10) into (21) and carrying out inner integration along φ, with (17) taken into account, we obtain

$$M_x^- = 4h_0 b_0 \Phi^-(c_0), \qquad M_y^- = -4h_0 a_0 \Phi^-(c_0), \qquad M_z^- = 0 \tag{22}$$

Here

$$\Phi^-(c_0) = \sqrt{1-c_0^2}\int_{z_1}^{z_2} \frac{R(z)[z + R'(z)R(z)]}{1+[R'(z)]^2}\Big[\cos\varphi^* - \frac{1}{3}\cos^3\varphi^* -$$

$$- \sin\varphi^*\left(\frac{\pi}{2} - \varphi^*\right)\Big]dz - \pi c_0 \int_{z_2}^{z_3} \frac{R(z)R'(z)[z + R'(z)R(z)]}{1+[R'(z)]^2}\,dz$$

As can be easily seen, the function $\Phi(c_0)$ is expressed for the case of $c_0 > 0$. For $c_0 < 0$, the integration in the second term should be between the limits of z_4 to z_1. Consequently, in this case, there does not exist a single expression for the principal force and moment vectors of light pressure forces for an arbitrary sign of the quantity c_0. Only in the presence of double symmetry of the body can it be shown that

$$\Phi^-(c_0) = -\pi c_0 \int_0^{z_3} \frac{R'(z)R(z)}{1+[R'(z)]^2}[z + R'(z)R(z)]\,dz =$$

$$= -\pi c_0 \int_{z_4}^{0} \frac{R'(z)R(z)}{1+[R'(z)]^2}[z + R'(z)R(z)]\,dz \tag{23}$$

Thus, the projections of the principal moment of light pressure forces acting on a geometrically symmetrical body are

$$M_x = h_0 b_0 \Phi_0(c_0), \qquad M_y = -h_0 a_0 \Phi(c_0), \qquad M_z = 0 \tag{24}$$

or in vector form

$$\mathbf{M} = h_0\Phi(c_0)[\tau\times\mathbf{k}], \quad \Phi(c_0) = (1-\varepsilon)\Phi^+(c_0) + 4\varepsilon\Phi^-(c_0) \tag{25}$$

Here k is the unit vector along the axial direction of the body z.

The paper [2] derived an analogous formula for the moment of light forces without any preliminary expositions, and the method for computation of function similar to $\Phi(c_0)$, is not concretely defined.

The function $\Phi(c_0)$ will be called the determining function

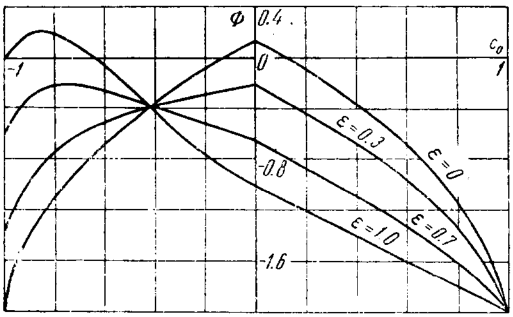

Fig. 2

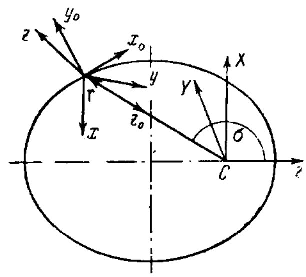

Fig. 3

since, as it will be shown in the following, its form will determine the stability of the motions being investigated.

In deriving Formulas (20) and (22), the location of the origin of the coordinates, i.e. the point about which the moment is computed, was not in any way stipulated. It will be stipulated now that the origin of the x, y, z axes coincides with the inertia center of the body, and that the axes are the principal central axes of inertia. If for any reason it is more convenient to compute the moment about another point on the axis of symmetry of the body, then, knowing the expressions for the projections of the principal force and principal moment, it is always possible to transfer the moment to the inertia center of the body. Such a transfer does not alter the general form of Formulas (24) and (25). For example, Fig.2 shows a plot of the function $\Phi(c_0)/(^3/_{16}\pi R^3)$ for a hemisphere of radius R for different values of the reflection coefficient ε , and where the origin of the coordinates is assumed to be the center of inertia of the hemisphere. Such a plot shows that the form of the function $\Phi(c_0)$ can be quite complicated.

Let us consider now the influence of the light pressure moment upon the motion of an artificial satellite of the sun about its center of inertia. In all subsequent investigations, the motion of the center of inertia itself is assumed known, i.e. we consider the problem of the motion of an artificial satellite in a field of given forces.

We will consider the following coordinate systems (Fig.3).

1. A fixed system of coordinates X, Y, Z with the origin at the center of the sun which is the focus of the elliptic orbit of the satellite. The Z-axis is directed along the radius vector to the orbit perihelion, the X-axis is parallel to the orbit perihelion tangent, the Y-axis is perpendicular to the plane of the orbit.

2. The orbital system of coordinates x_0, y_0, z_0 with the origin at the inertia center of the satellite. The z_0-axis is directed along the heliocentric vertical of the satellite inertia center, y_0 is perpendicular to the orbit plane and along the Y-axis, the x_0-axis completes the right-handed orthogonal coordinate system.

3. The x, y, z set of axes is the satellite body fixed system of coordinates. It will be assumed that this is the principal set of axes, that the satellite is geometrically symmetrical and that the z-axis is the axis of symmetry.

The direction cosines relating the introduced coordinate systems are defined as follows

(26)

	X	Y	Z
x	α_1	β_1	γ_1
y	α_2	β_2	γ_2
z	α_3	β_3	γ_3

	x_0	y_0	z_0
x	α	$\beta = \beta_1$	γ
y	α'	$\beta' = \beta_2$	γ'
z	α''	$\beta'' = \beta_3$	γ''

	X	Y	Z
x_0	$-\cos\sigma$	0	$\sin\sigma$
y_0	0	1	0
z_0	$-\sin\sigma$	0	$-\cos\sigma$

Taking into account the introduced assumptions, the equations of motion for the satellite about its center of mass are given in the form of the Euler equations

$$A\frac{dp}{dt} + (C - B)\,qr = h_0\gamma'\Phi(\gamma''), \qquad B\frac{dq}{dt} + (A - C)\,pr = -h_0\gamma\Phi(\gamma'') \tag{27}$$

$$C\frac{dr}{dt} + (B - A)\,pq = 0$$

Here A, B, C are the moments of inertia of the satellite relative to the x, y, z axes respectively; p, q, r are the projections of the absolute angular velocity of the satellite upon these axes. Equations (27) are augmented by the kinematical relationships

$$\begin{aligned}
&\frac{d\gamma}{dt} = r\gamma' - q\gamma'' + \omega\alpha, && \frac{d\alpha}{dt} = r\alpha' - q\alpha'' - \omega\gamma, && \frac{d\beta}{dt} = r\beta' - q\beta'' \\
&\frac{d\gamma'}{dt} = p\gamma'' - r\gamma + \omega\alpha', && \frac{d\alpha'}{dt} = p\alpha'' - r\alpha - \omega\gamma', && \frac{d\beta'}{dt} = p\beta'' - r\beta \\
&\frac{d\gamma''}{dt} = q\gamma - p\gamma' + \omega\alpha'', && \frac{d\alpha''}{dt} = q\alpha - p\alpha' - \omega\gamma'', && \frac{d\beta''}{dt} = q\beta - p\beta'
\end{aligned} \tag{28}$$

where ω is the orbital angular velocity of the satellite.

For the motion of the satellite along a circular orbit, Equations (27) and (28) possess the first integral

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + h_0\int \Phi(\gamma'')\,d\gamma'' - \omega\,(Ap\beta_1 + Bq\beta_2 + Cr\beta_3) = \mathrm{const} \tag{29}$$

Prior to studying the stability of the rotational motion of the satellite we will consider the influence of the light flow upon the motion of a rigid body relative to a fixed point at the center of inertia of the body. In this case, the first integral of motion is of the form

$$\frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + h_0\int \Phi(\gamma'')\,d\gamma'' = \mathrm{const} \tag{30}$$

This expression can be regarded as an energy integral since the first term represents the kinetic energy of the body. Let ϑ be the angle between the z-axis and the unit vector τ. The potential energy is then

$$\Pi = h_0\int \Phi(\gamma'')\,d\gamma'' \quad \text{or} \quad \Pi = -h_0\int \Phi(\cos\vartheta)\sin\vartheta\,d\vartheta \qquad (\gamma'' = \cos\vartheta) \tag{31}$$

As is known [3], the stability of equilibrium of a conservative system is assured if the potential energy of the system has a minimum at the equilibrium point. Finding the minimum of the function (31) by the usual methods we get

$$\frac{d\Pi}{d\vartheta} = -h_0\Phi(\cos\vartheta)\sin\vartheta = 0 \quad \text{for } \vartheta_1 = 0,\ \vartheta_2 = \pi \tag{32}$$

$$\frac{d^2\Pi}{d\vartheta^2} = -h_0\Phi(\cos\vartheta)\cos\vartheta + h_0\frac{d\Phi}{d\cos\vartheta}\sin^2\vartheta \tag{33}$$

Consequently, the equilibrium position corresponding to $\vartheta = 0$, $\cos\vartheta = 1$, is stable for $\Phi(1) < 0$, and the equilibrium position $\vartheta = \pi$, $\cos\vartheta = -1$ is stable for $\Phi(-1) > 0$. These conditions are not only sufficient but also necessary.

Thus, the geometrically symmetrical rigid body rotating about a fixed point coinciding with its inertia center and subject to the moment of light pressure forces necessarily has two positions of equilibrium corresponding to the coincidence of the axis of symmetry of the body with the direction to the light source. The stability or instability of the equilibrium position is determined by the sign of the function $\Phi(c_0)$ at the point of equilibrium.

As follows from (32), there are possible positions of equilibrium when $\Phi(\cos\vartheta) = 0$ other than the two indicated ones. The equilibrium positions for $\vartheta = 0$ and $\vartheta = \pi$ are termed basic since they always exist and for $\Phi(\cos\vartheta) = 0$ are called intermediate. As follows from (33), the intermediate position of equilibrium is stable if

$$\frac{d\Phi}{d\cos\vartheta} > 0 \quad \text{for } \Phi(\cos\vartheta) = 0 \tag{34}$$

The number (even or odd) as well as the interchange of stable and unstable intermediate positions of equilibrium is determined by the signs of the determining functions at the ends of their specified intervals, i.e. at the basic positions of equilibrium. Thus, the function $\Phi(c_0)$ fully determines the number, distribution and character of the positions of equilibrium for a rotating motion of a rigid body in a parallel light flow. In the cases when

$$\Phi(\pm 1) \equiv 0 \quad \text{or} \quad d\Phi/d\cos\vartheta \equiv 0, \;\; \Phi\cos\vartheta = 0$$

the minima and maxima of $\Phi(c_0)$, and consequently the stability of the position of equilibrium should be determined on the basis of higher derivatives at the extremum points.

As an example, let us turn to Fig.2 which shows the plot of the function $\Phi(c_0)$ for a hemisphere. For $\epsilon = 0$ the basic position of equilibrium for $\vartheta = 0$ is stable, then follows an unstable intermediate, a stable intermediate, and an unstable basic position of equilibrium. For $\epsilon = 0.3$ and $\epsilon = 0.7$ there exist only two basic positions of equilibrium one of which (for $\vartheta = 0$) is stable, and the other one unstable. Finally, for $\epsilon = 1$ the basic position of equilibrium $\vartheta = 0$ is stable, then follows an unstable intermediate position, and possibly, a stable basic position of equilibrium.

Let us review the stability conditions for the basic positions of equilibrium of a body with a completely absorbing ($\epsilon = 0$) or a completely reflecting ($\epsilon = 1$) surface.

In the first case, the projections of the moment of forces and the determining function are of the form (20) whence it follows that for $c_0 = \pm 1$ and, consequently, $z_1 = z_2 = z^*$

$$\Phi^+(\pm 1) = \pi z^* R^{*2} \tag{35}$$

This means that the stability conditions for the basic positions of equilibrium are of the form

$$z^* < 0 \quad \text{for } \vartheta = 0, \qquad z^* > 0 \quad \text{for } \vartheta = \pi \tag{36}$$

Condition (36) are contradictory and, consequently, one of the basic positions of equilibrium is stable and the other unstable. It can be shown that in the case being considered z^* is the coordinate of the center of pressure, i.e. of such a point about which the moment of the acting forces is zero. Consequently, the basic positions of equilibrium are stable if the center of pressure is behind the center of mass.

Let us find now the value of the determining function $\Phi^-(c_0)$ for $c = \pm 1$. According to (22)

$$\Phi^-(1) = -\pi \int_{z^*}^{z_3} \frac{R(z)\,R'(z)\,[z + R'(z)\,R(z)]}{1 + [R'(z)]^2}\,dz \tag{37}$$

Since $R'(z) < 0$ for $z^* < z < z_3$, then the sufficient condition for the position of equilibrium is the inequality

$$z + R'(z)\,R(z) < 0 \quad \text{for } z^* < z < z_3 \tag{38}$$

Analogously, the sufficient condition for $\vartheta = \pi$ is of the form

$$z + R'(z)\,R(z) > 0 \quad \text{for} \quad z_4 < z < z^* \tag{39}$$

If the function $z + R'(z)\,R(z)$ changes the sign in the considered range of the argument variation, then the integrals must be evaluated according to (37), and the usual conditions of stability reviewed. Finally, for an arbitrary reflection coefficient the stability conditions for the basic positions of equilibrium are of the form

$$(1 - \varepsilon)\, z^* R^{*2} - 4\varepsilon \int_{z^*}^{z_3} \frac{R(z)\,R'(z)\,[z + R'(z)\,R(z)]}{1 + [R'(z)]^2}\, dz < 0 \quad \text{for } \vartheta = 0$$

$$(1 - \varepsilon)\, z^* R^{*2} + 4\varepsilon \int_{z_4}^{z^*} \frac{R(z)\,R'(z)\,[z + R'(z)\,R(z)]}{1 + [R'(z)]^2}\, dz > 0 \quad \text{for } \vartheta = \pi \tag{40}$$

The presented method is also applicable for plane, cylindrical as well as conical sections of the body's surface. In these cases the function ~ must be computed by taking into consideration the specific form of the body surface.

Let us return to the consideration of the rotational motion of the artificial satellite of the sun. Equations (27) and (28) have the following particular solutions

$$\begin{gathered} p = r = 0; \quad q = \omega; \quad \gamma' = \gamma = 0, \quad \gamma'' = \pm 1 \\ \alpha'' = \alpha' = 0, \quad \alpha = \mp 1; \qquad \beta = \beta'' = 0, \quad \beta' = 1 \end{gathered} \tag{41}$$

which correspond to the rotational motion of the satellite about the center of inertia in such a way that the x, y, z axes "track" the orbital system of coordinates, and in particular, the symmetry axis of the satellite is at all times coincident with the direction to the sun. We will investigate the stability of such motion by representing p, q, r in the form

$$p = p_* + \omega\beta_1, \qquad q = q_* + \omega\beta_2, \qquad r = r_* + \omega\beta_3 \tag{42}$$

The quantities p_*, q_*, r_* represent the axial projections of the satellite angular velocity in the perturbed state. Transforming the integral (29) we obtain

$$\frac{1}{2}(Ap_*^2 + Bq_*^2 + Cr_*^2) + h_0 \int \Phi(\gamma'')\, d\gamma'' - \frac{1}{2}\omega^2 (A\beta_1^2 + B\beta_2^2 + B\beta_3^2) = \text{const} \tag{43}$$

Expanding $\Phi(\gamma'')$ into a Taylor series about $\gamma'' = \pm 1$ and taking into consideration in (43) only the quantity $\Phi(\pm 1)$, we find that for the coordinates in the perturbed state there exists the integral $(\Delta\gamma'' > 0)$ (44)

$$\tfrac{1}{2}(Ap_*^2 + Bq_*^2 + Cr_*^2) \mp h_0 \Phi(+1)\Delta\gamma'' + \tfrac{1}{2}\omega^2\, [(B - A)\beta_1^2 + (B - C)\beta_3^2] = \text{const}$$

Consequently, the unperturbed motion corresponding to $\gamma'' = +1$ is stable for

$$\Phi(+1) < 0, \quad B > A, \ B > C \tag{45}$$

and corresponding to $\gamma'' = -1$ is stable for

$$\Phi(-1) > 0, \ B > A, \ B > C \tag{46}$$

since in fulfilling conditions (45) and (46), the integral (44) is a sign-determined positive function of the coordinates of the perturbed motion.

Thus, there are two steady atate rotational motions corresponding to the two basic positions of equilibrium in the orbital system of coordinates in the case of the geometrically symmetrical artificial satellite of the sun in a circular orbit. The stability of these motions depends,as before, upon the sign of the determining function at the point considered. However, in addition, the stability conditions must be augmented by the relationships among the dynamical characteristics which are the moments of inertia of the satellite. These additional inequalities are stipulated by the orbital motion of the satellite's center of inertia. We will make a few remarks.

1. It can be shown that the inequalities (45) and (46) will be not only

sufficient but also necessary conditions of stability.

2. It can be shown that the last term in (44) is stipulated by the field of the centrifugal forces originating from the satellite's motion in a circular orbit.

3. In addition to the particular solutions (41) which always exist regardless of the form of the function $\Phi(\gamma'')$, the following particular solution is possible

$$p = r = 0, \quad q = \omega; \quad \Phi(\gamma_x'') = 0 \tag{47}$$

$$\alpha_x = \gamma_x'', \quad \alpha_x'' = -\gamma_x; \quad \alpha_x' = \gamma_x' = 0, \quad \beta = \beta'' = 0, \quad \beta' = 1$$

Solution (47) corresponds to the "tracking" by the satellite of the orbital system of coordinates in which the symmetry axis of the satellite is at an angle χ relative to the direction of the light source which is defined by the relations

$$\alpha_x = \gamma_x'' = \cos\chi, \qquad \alpha_x'' = \sin\chi \tag{48}$$

The conditions of stability for such a motion also consist of two groups of inequalities which are of the form

$$d\Phi/d\gamma'' > 0 \quad \text{for} \quad \Phi(\gamma'') = 0; \qquad B > A, \qquad B > C \tag{49}$$

4. In investigating the motion of an artificial satellite of the sun in the field of light pressure forces, the sun's gravitational force was neglected. As is known, in a circular orbit the moment of the gravitational forces is proportional to ω^2, where ω is the orbital angular velocity of the satellite. The potential energy of the centrifugal forces in (44) is also proportional to ω^2. Therefore, to the same order of accuracy that the moment of the gravitational forces is small compared to the moment of light pressure forces, the potential energy of the centrifugal force field may be regarded small compared to the potential energy of the light pressure force field.

5. Formulas for the projection of the moment of light pressure forces acting upon a geometrically symmetrical satellite of the sun, and the first integral resulting from the integration of the equations of motion are similar in structure to the expressions for the moment and the first integral of motion in the case of a satellite rotating about its center of inertia in a central field of gravitational forces [4 and 5]. Although the consideration of a physical nature of these forces reveals more differences than similarities, from the mathematical point of view, the rotational motion of the satellite in the light pressure force field is a more general case of the rotating motion of the satellite in a central gravitational force field.

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